Towards a Spectral Approach for the Design of Self-Synchronizing Stream-Ciphers

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Abstract. This paper addresses the problem of self-synchronization of a class of dynamical systems involving Boolean functions. We motivate the work in the context of cryptography with the perspective of designing self-synchronizing stream ciphers and assessing their efficiency in terms of security. The self-synchronization is tackled through the notion of influence of variables. We propose a spectral characterization of the self-synchronization property through the Walsh transform. We discuss two kinds of self-synchronization: the finite time and the statistical one.

1 Introduction

Stream-ciphers are cryptosystems well suited when the data to be encrypted are generated sequentially or when the encryption speed is a concern. Indeed, their simple structure usually allows higher throughput than block ciphers. Unlike block ciphers, they involve a time varying transformation applied to the plain-text giving rise to the interesting property that two identical plain-texts may have different cipher-texts. An important requirement for proper decryption is to guarantee that the decipher is well synchronized with the cipher. This is often ensured by resorting to synchronization protocols which introduce overheads in the stream packets conveyed through the channel. Such an approach applied for the so-called synchronous stream ciphers. A special class of stream-ciphers has the interesting property of self-synchronization. By self-synchronization, it is meant an intrinsic (also called structural) synchronization between the decipher and the cipher rendering the synchronization protocols useless. They are called self-synchronizing stream-ciphers (SSSC for short). And yet, it must be stressed that few attention has been paid on them. The reader may refer to [1] for details about the design of SSSC.

In this paper, we focus on the self-synchronization property for systems involving Boolean functions [2][3] having in mind the cryptographic context and more specifically the use of self-synchronizing stream ciphers. The self-synchronization is tackled through the notion of influence of variables which is developed herein. Then, we propose a spectral characterization of the self-synchronization property. The layout is the following: Section 2 is devoted to the problem statement. Canonical equations of SSSC are recalled. In Section 3, we expose the main results which

essentially consist in a spectral characterization of the self-synchronization property and the notion of influence of variables.

2 Problem Statement

Concealment in stream-ciphers is usually done by performing the exclusive-OR (denoted by $\oplus$) between the successive plain-text bits (denoted by p_k) and the bits of a key-stream (denoted by K_k). The index k stands for the discrete-time. The cipher and the decipher can be respectively modeled as dynamical systems governed by the sets of equations (1) and (2).

$$\begin{cases} x_k &= f(c_k, x_{k-1}) \\ K_k &= g(x_k, c_{k-1}) \\ c_k &= K_k \oplus p_k \end{cases} \qquad \text{(cipher equation)} \tag{1}$$

$$\begin{cases} \widehat{x}_k &= f(c_k, \widehat{x}_{k-1}) \\ \widehat{K}_k &= g(\widehat{x}_k, c_{k-1}) \\ \widehat{p}_k &= \widehat{K}_k \oplus c_k \end{cases} \qquad \text{(decipher equation)} \tag{2}$$

Let us call a (n, m)–function a function from the vector space $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$. x^k and $\widehat{x}_k$ are respectively the state vectors of the cipher and of the decipher. The next-state function f is an $(n+1, n)$ function depending on x_{k-1} and the cipher-text c_k. g is an $(n, 1)$–function which delivers the key-streams K_k or $\widehat{K}_k$. Finally, $\widehat{p}_k$ stands for the recovered plain-text. For a proper decryption of the cryptogram c_k, the states x_k and $\widehat{x}_k$ must coincide at each discrete-time k. Roughly speaking, the compound system (1) and (2) is self-synchronizing if the coincidence $x_k = \widehat{x}_k$ is achieved whatever are the initial states x_0 and $\widehat{x}_0$, possibly after a transient time. In this sense, the synchronization is an intrinsic (or structural) property of the system. It can easily be seen that this property depends exclusively on the function f. Now, let us formally define the self-synchronization property. We denote by (c) the sequence $(c_0, \ldots, c_i)$.

Definition 1 (Self-synchronizing sequence). *A sequence (c) is self-synchronizing for f if there exists an integer k_c so that for all initial state x_0 and $\widehat{x}_0$*

$$\forall k \geq k_c,\ x_k = \widehat{x}_k \tag{3}$$

Definition 2 (Finite time self-synchronization). *The system (1)-(2) is finite time self-synchronizing if the minimum value k_c is upper bounded over all possible sequences (c). The upper bound k_c is called the self-synchronization delay of f.*

Remark 1. If (c) is a random sequence, then k_c is a random variable. In such a case, it should be denoted K_c.

Definition 3 (Statistical self-synchronization). *The system (1)-(2) is statistically self-synchronizing if $\lim_{k \to +\infty} \text{Prob}(K_c \leq k) = 1$. K_c is called the random synchronization delay for the random sequence (c).*

3 Main Results

This section aims at stating a spectral characterization of the self-synchronizing property through the Walsh transform. Such a property is expressed in terms of influence of variables which is developed herein.

3.1 Self-Synchronization and Influence

The function f involved in (1)-(2) is considered as a mapping $f(c,x) : \mathbb{F}_2 \times \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n$. We define, for any positive integer i, the $(n+i+1,n)$–function ϕ_i which is the iterated function of f. It expresses the internal state after $i+1$ iterations and thereby depends on the initial internal state and the cipher-text sequence. For $x_0 \in \mathbb{F}_2^n$ and $(c) = (c_0, \ldots, c_i) \in \mathbb{F}_2^{i+1}$, $\phi_i(c,x)$ is defined as:

$$\phi_i(c,x) = f(c_i, f(c_{i-1}, f(\cdots, f(c_0, x_0)\cdots))) \tag{4}$$

The self-synchronization property can be directly related to the notion of influence. Indeed, it is worth pointing out that (1)-(2) has the self-synchronization property if there exists an integer i so that ϕ_i does not depend on x_0 anymore.
Therefore, we have the following Proposition

Proposition 1. *The system* (1)-(2) *is self-synchronizing if and only if* x_0 *has no influence over* ϕ_i.

The function ϕ_i playing a central role and having in mind a spectral characterization of the influence, we must state some properties of the corresponding Walsh transform.
Let us denote denote by f^0 (respectively f^1) the (n,n)–function which is the restriction of f to the input $c = 0$ (respectively to $c = 1$). For a given fixed input sequence (c), we denote by ϕ_i^c the (n,n)–function that expresses the internal state after $i+1$ iterations: $\phi_i^c : x \longmapsto \phi_i(c,x)$. The spectrum of the function ϕ_i restricted to the input sequence (c) is the Walsh matrix $W_{\phi_i^c}$ of ϕ_i^c.

Proposition 2. *The Walsh matrix of* ϕ_i^c *is*

$$W_{\phi_i^c} = \frac{1}{2^{n+i}} W_{f^{c_i}} W_{f^{c_{i-1}}} \times \cdots \times W_{f^{c_0}}. \tag{5}$$

The expression of the spectrum of ϕ_i can be expressed with the spectrum of f. For two vectors $u = (u_0, \ldots u_i)$ and $v = (v_0, \ldots, v_{n-1})$, their concatenation, denoted $u|v$ is by definition the $(n+i+1)$–dimensional vector $u|v = (u_0, \ldots n_i, v_0, \ldots, v_{n-1})$. We define the operator τ which transforms a binary vector $z = (z_0, \cdots, z_\ell)$ to the integer $\tau(z) = \sum_{i=0}^{\ell} 2^{\ell-i} z_i$.

Proposition 3. *The entry of the* $2^n \times 2^{n+i+1}$ *dimensional Walsh matrix of the iterated function* ϕ_i *is*

$$w_{t,\tau(u|v)}^{\phi_i} = \sum_{c \in \mathbb{F}_2^{i+1}} (-1)^{u \cdot c} w_{t,v}^{\phi_i^c}. \tag{6}$$

3.2 Spectral Characterization of the Influence

Throughout the literature, see in particular [4] and [5], the influence of a set S of variables over a function is often defined as the probability that there exists at least one way to change the value of the function by changing the value of the variables in the set, the variables not in the set being chosen at random. The fact that this definition has no usable spectral expression has a major drawback. Indeed, it does not take into account the number of possibilities of choosing the variables for changing the value of the function. We therefore rather introduce another definition of the influence that takes this point into account. A spectral characterization is more appropriate to this end.

Definition 4. *The influence on a function f of a set S of variables is the mean of the probabilities that $f(x)$ changes when x is randomly chosen, the mean being computed for all possibles changes of variables not in the set S.*

Recall that the support of a vector $u \in \mathbb{F}_2^n$ is by definition $\mathrm{supp}(u) = \{i \in \{1, \cdots, n\} | u_i \neq 0\}$.

Definition 5. *Let f be a $(n,1)$–function of a vector $x \in \mathbb{F}_2^n$ and S a set of ℓ components of x. The influence $I_f(S)$ is:*

$$I_f(S) = \frac{1}{2^n(2^\ell - 1)} \sum_{x \in \mathbb{F}_2^n} \sum_{u \in \mathbb{F}_2^n / u \neq 0,\ \mathrm{supp}(u) \subset S} [f(x) \oplus f(x \oplus u)] \tag{7}$$

It can be shown that $I_f(S)$ can be expressed by means of $\widehat{f_\chi}$ which is the Walsh transform of f. Notice that the correspondence between $\widehat{f_\chi}$ and the Walsh matrix fulfills $w_{1,\tau(v)} = \widehat{f_\chi}(v)$.

Proposition 4. *Let f be a $(n,1)$–function of a vector $x \in \mathbb{F}_2^n$ and S a set of ℓ components of x,*

$$I_f(S) = \frac{2^{\ell-1}}{2^{2n}(2^\ell - 1)} \sum_{v \in \mathbb{F}_2^n / \,\mathrm{supp}(v) \cap S \neq \emptyset} \widehat{f_\chi}^2(v) \tag{8}$$

We extend Definition 5 to a vectorial Boolean function that is to a (n,m)–function with $m \geq 1$.

Definition 6. *The influence of a set of variables S over a vectorial Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is*

$$I_f(S) = \frac{1}{m} \sum_{j=1}^{m} I_{f_j}(S) \tag{9}$$

where f_j is the j^{th} component function of f.

3.3 Spectral Characterization of the Self-Synchronization Property

In this section, we characterize the self-synchronizing property based of the results stated above.

Finite Time Self-Synchronization We recall that the self-synchronization property is related to the existence of an integer i so that $\phi_i(c, x_0)$ does not depend on x_0 or equivalently to the fact that the initial state x_0 has no influence over ϕ_i. By virtue of (8) and (9), the influence is proportional to the sum of some squared Walsh coefficients. This yields the following proposition.

Proposition 5. *Let S_{x_0} be the set of n components of x_0 the initial state of the functions ϕ_i. The system (1)-(2) is finite time self-synchronizing if and only if there exists a corresponding iterated function ϕ_i so that,*

$$\forall v \in \mathbb{F}_2^{n+i+1} / \operatorname{supp}(v) \cap S_{x_0} \neq \emptyset,\ \forall j \in [0, 2^{n-1} - 1],\ w^{\phi_i}_{j,\tau(v)} = 0 \tag{10}$$

Statistical Self-Synchronization We now consider the case of the statistical self-synchronization property. If a system is statistically self-synchronizing, there exists at least one self-synchronizing sequence. It is possible to characterize such a sequence using the following proposition.

Proposition 6. *The sequence (c) of length $i+1$ is a self-synchronizing sequence if and only if the Walsh matrix of the function ϕ_i^c is of the form*

$$W_{\phi_i^c} = \begin{pmatrix} 2^n & 0 & \cdots & 0 \\ \pm 2^n & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \pm 2^n & 0 & \cdots & 0 \end{pmatrix} \tag{11}$$

4 Concluding Remarks

We have focus on the self-synchronization property for systems involving Boolean functions. It is motivated by the fact that such a property is central when addressing cryptographic applications involving the special class of stream ciphers, namely the self-synchronizing ones. We expect that this work constitutes a first step towards a systematic methodology for the design of such ciphers. Indeed, the design amounts to finding suitable Boolean functions f so that the iterated function fulfills the spectral conditions provided in this note.

References

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