Asymptotic nonlinearity of Boolean functions

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Outline

1. Boolean functions
2. Higher order nonlinearity of Boolean functions
3. Nonlinearity of Vectorial Boolean functions
4. Resistance against linear cryptanalysis
5. Conclusion

Boolean functions

- Let m be a positive integer and $q = 2^m$.
- A **Boolean function** with m variables is a map from the space $V_m = \mathbb{F}_2^m$ into $\mathbb{F}_2$.
- A Boolean function is **linear** if it is a linear form on the vector space $\mathbb{F}_2^m$.
- It is **affine** if it is equal to a linear function up to a constant.

Cryptanalysis

- Booleaan functions are used to build cryptosystems, block ciphers or stream ciphers.

- The existence of affine approximations of the Boolean functions involved in a cryptosystem allows in various situations to build attacks on this system.

- It consists in

  simplifying the enciphering algorithm by a **linear approximation.**

- Therefore a function f is **the more resistant** to this attack that f is **distinct from a linear mapping.**

Non-linearity

We call **non-linearity** of a Boolean function $f : V_m \longrightarrow \mathbb{F}_2$
the distance from f to the set of affine functions with m variables:

$$nl(f) = \min_{h \text{ affine}} d(f, h)$$

where d is the Hamming distance.

The non-linearity is equal to $$nl(f) = 2^{m-1} - \frac{1}{2} S(f)$$

where

$$S(f) = \sup_{v \in V_m} \left| \sum_{x \in V_m} (-1)^{(f(x)+v \cdot x)} \right|$$

and $v \cdot x$ denote the usual scalar product in V_m.

$S(f)$ is **the spectral amplitude** of the Boolean function f.

Inequalities on the nonlinearity

$$2^{m/2} \quad \underset{\substack{\uparrow \\ \text{Parseval}}}{\leq} \quad 2^m - 2nl(f) = S(f) = \sup_{v \in V_m} \left| \sum_{x \in V_m} (-1)^{(f(x)+v \cdot x)} \right| \quad \underset{\substack{\uparrow \\ \text{clear}}}{\leq} \quad 2^m$$

For an even dimension m: bent functions reach the lower bound $2^{m/2}$.

For odd m: $2^{m/2}\sqrt{2}$ has been a long time the only known lower bound of the spectral amplitude $S(f)$.

Improvements of the bound by Patterson and Wiedemann and more recently, by Kavut, Maitra and Yücel have led to a conjecture:

$$\inf_f S(f) \sim 2^{m/2}$$

Boolean functions in cryptography

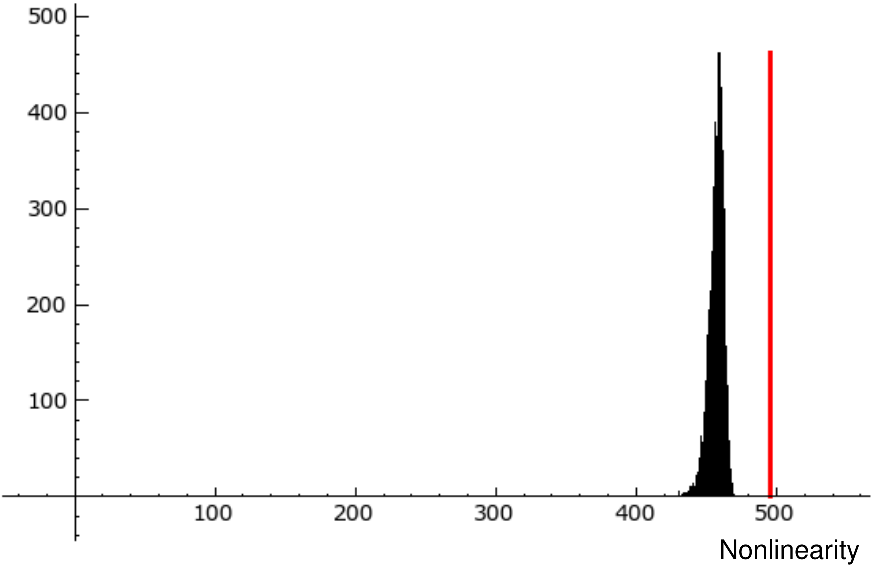
For security reasons functions need to have properties like

- high nonlinearity
- balancedness
- high algebraic degree,
- High algebraic immunity,
- ...

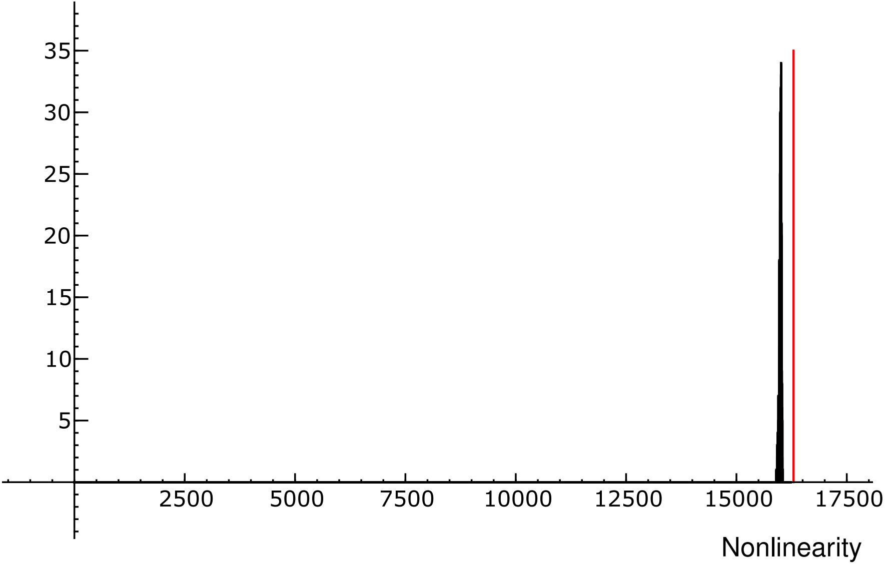
It is necessary to have the possibility of choosing among many Boolean functions,

- not only bent functions,
- but also functions which are **close to be bent,**

Distribution of the nonlinearity for $m = 10$



Distribution of the nonlinearity for $m = 15$



Distribution of the nonlinearity of the Boolean functions

Theorem (Olejar, Stanek, Carlet, FR)

The probability that

$$a\sqrt{2q\log q} < q - 2nl(f) = S(f) < b\sqrt{2q\log q}$$

tends to 1 as m goes to infinity for $0 < a < 1 < b$.

If f is a Boolean function, then, almost surely:

$$\lim_{m\to\infty} \frac{S(f)}{\sqrt{2q\log q}} = 1$$

Cryptanalysis of order r

- The cryptanalysis of order r consists in **simplify the enciphering algorithm** by making an **approximation** by the set of all functions whose algebraic degrees do not exceed r.

- Therefore a function f is **the more resistant** to that this attack that f is **distinct from a mapping of order r.**
- The nonlinearity of order r generalizes the usual nonlinearity.

  For a given function f, it is its Hamming distance to the set of all r-order functions
- Let $NL_r(f)$ denote the r-th order nonlinearity of f, we have

$$NL_r(f) = \min_{g \in \mathrm{RM(r,n)}} d_H(f, g).$$

Higher order nonlinearity of Boolean functions

Very little is known on $nl_r(f)$ for $r > 1$.

To be able to compare with the preceding theorem, we define the spectral amplitude of a Boolean function f the integer $S_r(f)$ such that

$$nl_r(f) = 2^{m-1} - \frac{1}{2} S_r(f).$$

Theorem (C. Carlet and S. Mesnager)

The minimum possible spectral amplitude of order r of Boolean functions, is bounded from below by $\sqrt{15}(1+\sqrt{2})^{r-2} \times 2^{m/2+1} + O(m^{r-2})$.

Higher order nonlinearity of Boolean functions

Asymptotically, C. Carlet proved that almost all Boolean functions have high r-th order nonlinearities, or low r-th order spectral amplitude.

S. Dib, K-U. Schmidt proved that this was the exact bound.

Theorem (C. Carlet, S. Dib, K-U. Schmidt)

The density of the set of functions satisfying

$$a2^{\frac{m+1}{2}}\sqrt{\binom{m}{r}\log 2} < 2^m - 2nl_r(f) = S_r(f) < b2^{\frac{m+1}{2}}\sqrt{\binom{m}{r}\log 2}$$

tends to 1 when m tends to infinity, if $0 < a < 1 < b$.

If f is a Boolean function, then, almost surely:

$$\lim_{m\to\infty}\frac{S(f)}{2^{\frac{m+1}{2}}\sqrt{\binom{m}{r}\log 2}} = 1$$

Vectorial Boolean functions - S-boxes

The linear cryptanalysis exploits **nonuniform statistical behaviors** in the process of encryption.

It consists in **simplifying the encryption algorithm** by making a **linear approximation**.

Therefore a function F is **the more resistant** to that this attack that F is **distinct from a linear mapping.**

Vectorial Boolean functions - S-boxes

A (m, n) vectorial Boolean function with m variables is a map from the space $V_m = \mathbb{F}_2^m$ into $V_n = \mathbb{F}_2^n$.

I define the component functions $u \cdot f$ as the functions $x \longmapsto (u \cdot f)(x) = u \cdot f(x)$ where $\cdot$ denote the usual scalar product of two elements of V_n.

Non-linearity

We call **non-linearity** of a vectorial Boolean function $f : V_m \longrightarrow V_n$ the minimum Hamming distance between all the component functions of f and all affine functions on m variables:

$$nl(f) = \min_{u \in V_n^*} \min_{h \text{ affine}} d(u \cdot f, h)$$

where d is the Hamming distance.

The non-linearity is equal to $nl(f) = 2^{m-1} - \frac{1}{2} S(f)$

where $S(f) = \sup_{u \in V_n^*} \sup_{v \in V_m} \left| \sum_{x \in V_m} (-1)^{(u \cdot f(x) + v \cdot x)} \right|$

Vectorial Boolean functions - S-boxes

Results by Nyberg

Theorem (Nyberg)

The minimal spectral amplitude of a vectorial function $f : \mathbb{F}_{2^m} \longrightarrow \mathbb{F}_{2^n}$ *such that* $n \leq m - 1$ *is such that*

$$S(f) \geq 2^{m/2}$$

This bound can be achieved with equality only if m *is even and* $n \leq m/2$ *by the so-called bent functions.*

Results by Chabaud-Vaudenay

Theorem (Chabaud-Vaudenay)

The minimal spectral amplitude of a vectorial function $\mathbb{F}_{2^m} \longrightarrow \mathbb{F}_{2^m}$ *is* $2^{\frac{m+1}{2}}$.

The functions reaching this bound are called **almost bent.** *They exists when* m *is odd.*

Vectorial Boolean functions - S-boxes

Theorem (S. Dib)

If f is a vectorial function $\mathbb{F}_q \longrightarrow \mathbb{F}_q$, then, almost surely: the probability that

$$2a\sqrt{q\log q} < q - 2nl(f) = S(f) < 2b\sqrt{q\log q}$$

tends to 1 as m goes to infinity for $0 < a < 1 < b$.

If f is a vectorial function $\mathbb{F}_q \longrightarrow \mathbb{F}_q$, then, a.s.:

$$\lim_{m\to\infty} \frac{S(f)}{2\sqrt{q\log q}} = 1.$$

Vectorial Boolean functions - S-boxes

Theorem (S. Dib)

If f is a vectorial function $\mathbb{F}_2^m \longrightarrow \mathbb{F}_2^n$, then, almost surely: the probability that

$$S(f) < b\sqrt{2^{m+1}(m+n)\log 2}$$

tends to 1 as m goes to infinity for $1 < b$.

Theorem (S. Dib)

If f is a vectorial function $\mathbb{F}_2^m \longrightarrow \mathbb{F}_2^n$, and $m \geq n$ then, almost surely: the probability that

$$a\sqrt{2^{m+1}(m+n)\log 2} < S(f)$$

tends to 1 as m goes to infinity for $0 < a < 1$.

Resistance against linear cryptanalysis

Let an r round cipher with

- $X \in \mathbb{F}_2^m$, the plain text,
- $K \in \mathbb{F}_2^\ell$ the key
- $Y(X, K) \in \mathbb{F}_2^m$ a function of X and K.

where all are random variables.

$$x_1 \xrightarrow{F_{K_1}} x_2 \xrightarrow{F_{K_2}} \dots \xrightarrow{F_{K_{r-2}}} x_{r-1} \xrightarrow{F_{K_{r-1}}} Y(X, K)$$

Let $a \in \mathbb{F}_2^m$, $b \in \mathbb{F}_2^m$ be linear masks.

$$a \cdot X = b \cdot Y(X, K) \ ?$$

Theorem (Nyberg)

In a DES-like cipher with more than 4 rounds, independent round keys and uniformly random plaintext and f be the S-box.

$$2^{-\ell} \sum_{K \in \mathbb{F}_2^\ell} \left(P_X\left(a \cdot X = b \cdot Y(X, K)\right) - \frac{1}{2}\right)^2 \leq 2^{-4m-1} S(f)^4$$

Example

$$2^{-\ell} \sum_{K \in \mathbb{F}_2^{\ell}} \left(P_X \left(a \cdot X = b \cdot Y(X, K) \right) - \frac{1}{2} \right)^2 \leq 2^{-4m-1} S(f)^4$$

- Let we consider 2 ciphers
  - Let a cipher on $\mathbb{F}_2^m$ with f be an almost bent function.
  - Let a cipher on $\mathbb{F}_2^{m'}$ with f' a function $\mathbb{F}_2^{m'} \longrightarrow \mathbb{F}_2^{m'}$ such that $S(f') \simeq 2\sqrt{2^{m'} m' \log 2}$.
- Then they give the same bound on probability of breaking the cipher if

$$m = m' - log_2(m') - 0.47$$

- Let $m' = 136$ for random function, then $m = 128$ for almost bent function.

Conclusion

- We have been interested in classifying the Boolean functions according to the nonlinearity.
- We found a concentration point for the nonlinearity of random functions in the case of
  - Boolean functions with r^{th} order nonlinearity
  - Vectorial Boolean functions
- We found that it is close to the maximum nonlinearity in these cases
- You don't lose very much by replacing an almost bent function by a random function

- To be done

  to find bounds for $\sqrt{q} \leq S(f) \leq \sqrt{2q \log q}$.

Thank you