# Polar Grassmann codes of orthogonal type

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# Grassmannians

$V := V(m, q), \quad 1 \leq k < m$

$\mathcal{G}_{m,k}$: $k$–Grassmannian of $\mathrm{PG}(V)$

* Points of $\mathcal{G}_{m,k}$: $k$–dimensional subspaces of $V$.
* Lines of $\mathcal{G}_{m,k}$: sets $l_{X,Y} := \{Z : X < Z < Y\}$ with $\dim(X) = k-1$, $\dim(Y) = k+1$.

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Grassmann or Plücker embedding of $\mathcal{G}_{m,k}$

$$e_k : \mathcal{G}_{m,k} \to \mathrm{PG}(\bigwedge^k V)$$
$$\langle v_1, \ldots, v_k \rangle \to \langle v_1 \wedge v_2 \wedge \cdots \wedge v_k \rangle$$

* $\dim(e_k) = \binom{m}{k}$

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  $k = n$: sets $l_X := \{Z \colon X < Z < X^\perp\}$ with
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If $k < n$ then $\Delta_{n,k} \subseteq \mathcal{G}_{2n+1,k}$

Grassmann or Plücker embedding of $\Delta_{n,k}$

$$\varepsilon_k : \begin{cases} \Delta_{n,k} \to \mathrm{PG}(W_k) \subseteq \mathrm{PG}(\bigwedge^k V) \\ \langle v_1, \dots, v_k \rangle \to \langle v_1 \wedge v_2 \wedge \cdots \wedge v_k \rangle \end{cases}$$

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$k = n$ : lines of $\Delta_{n,n}$ are mapped onto non singular conics of $\mathrm{PG}(W_n)$.

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$\varepsilon^{spin} : \Delta_{n,n} \to \mathrm{PG}(2^n - 1, q)$ : spin embedding

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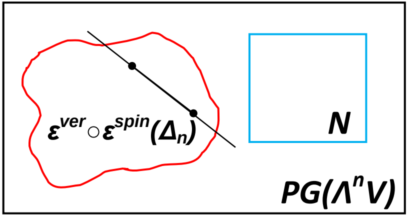


$$(\varepsilon^{ver} \circ \varepsilon^{spin})/\mathcal{N} : \Delta_{n,n} \to \mathrm{PG}(\bigwedge^n V/\mathcal{N})$$

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**Theorem**

*Any semilinear collineation of $\mathrm{P\Gamma L}(K, q)$ stabilizing $\Omega$ induces automorphisms of $\mathcal{C}(\Omega)$.*

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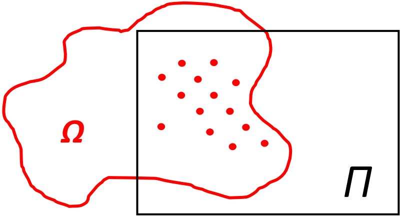

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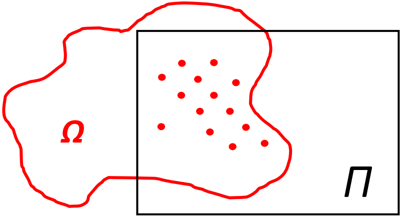


The study of the weights of $\mathcal{C}(\Omega)$ is equivalent to the study of the hyperplane sections of $\Omega$.

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**Theorem [Nogin, 1996]**

The parameters of $\mathcal{C}(\mathcal{G}_{m,k})$ are:

$$N = \begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})},$$

$$K = \binom{m}{k}, \qquad d_{\min} = q^{(m-k)k}.$$

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$k$–multilinear alternating forms on $V$ $\leftrightarrow$ Hyperplanes of $\bigwedge^k V$

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## Remark

- *Minimum weight codewords in a Grassmann code correspond to k–multilinear alternating forms with a maximum number of totally isotropic spaces.*
- *When $k = 2$ these are non–null forms with maximum radical.*

**Definition**

- $\Delta_{n,k}$: Orthogonal Grassmannian
- $\mathcal{C}(\Delta_{n,k}) := \mathcal{C}(\varepsilon_k(\Delta_{n,k}))$: Orthogonal Grassmann code, determined by $\varepsilon_k(\Delta_{n,k})$.

* I.C., Luca Giuzzi, *Codes and caps from Orthogonal Grassmannians*, Finite Fields Appl. **24** (2013), 148-169.

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- Interesting geometry

# Orthogonal Grassmann codes: previous results

**Theorem [I.C., Luca Giuzzi]**

For $1 \leq k < n$, the parameters of $\mathcal{C}(\Delta_{n,k})$ are

$$N = \prod_{i=0}^{k-1} \frac{q^{2(n-i)} - 1}{q^{i+1} - 1}, \qquad K = \begin{cases} \binom{2n+1}{k} & \text{for } q \text{ odd} \\ \binom{2n+1}{k} - \binom{2n+1}{k-2} & \text{for } q \text{ even} \end{cases}$$

$$d \geq (q+1)(q^{k(n-k)} - 1) + 1.$$

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## Theorem

$MAut(\mathcal{C}(\Delta_{n,k})) \cong \Gamma O(2n+1, q)$.

**Theorem [I.C., Luca Giuzzi, 2013]**

The code $\mathcal{C}(\Delta_{2,2})$ arising from $\varepsilon_2(\Delta_{2,2})$ has parameters

$$N = (q^2+1)(q+1), \qquad K = \begin{cases} 10 & \text{for } q \text{ odd} \\ 9 & \text{for } q \text{ even} \end{cases}$$

$$d = q^2(q-1).$$

**Theorem [I.C., Luca Giuzzi, 2013]**

The code $\mathcal{C}(\Delta_{3,3})$ arising from $\varepsilon_3(\Delta_{3,3})$ has parameters

$$\left.\begin{array}{r} N=(q^3+1)(q^2+1)(q+1), \quad K=35, \\ d=q^2(q-1)(q^3-1) \end{array}\right\} \quad \text{for } q \text{ odd}$$

and

$$\left.\begin{array}{r} N=(q^3+1)(q^2+1)(q+1), \quad K=28, \\ d=q^5(q-1) \end{array}\right\} \quad \text{for } q \text{ even.}$$

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  $\downarrow$

  maximum number of lines being simultaneously totally singular for the quadratic form $\eta$ (defining $\Delta_{n,2}$) and totally isotropic for a (degenerate) alternating form of $V$ (defining a hyperplane of $\bigwedge^2 V$).

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For $q$ odd the minimum distance of the codes $\mathcal{C}(\Delta_{n,2})$ is

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- Symplectic/Hermitian Grassmann codes