On the maximal number of points on singular curves over finite fields

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(Joint work with Yves Aubry)

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Notation

- $\mathbb{F}_q$ the finite field with q elements.
- With the word "curve" we will always refer to an absolutely irreducible projective algebraic curve.

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Let us denote by

$$N_q(g)$$

the maximal number of rational points over $\mathbb{F}_q$ that a curve of genus g can have. Clearly we have:

$$N_q(g) \leq q + 1 + g[2\sqrt{q}]$$

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If now we remove the hypothesis of smoothness for X, can we still say something about $\sharp X(\mathbb{F}_q)$?

Yes, but we have to introduce another invariant for X,

the arithmetic genus π.

To define π we have to recall some local properties of curves.

Points and local rings

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$$\mathcal{O}_Q := \{f \in \mathbb{F}_q(X) \mid f \text{ is regular at } Q\}.$$

$\mathcal{O}_Q$ is a local ring with maximal ideal

$$\mathcal{M}_Q := \{f \in \mathcal{O}_Q \mid f \text{ vanishes at } Q\}$$

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Fact: $\mathcal{O}_Q$ is integrally closed if and only if Q is a nonsingular point.

$$\Downarrow$$

X is smooth if and only if $\mathcal{O}_Q$ is integrally closed for every Q on X.

Normalization of a singular curve

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In particular X and $\tilde{X}$ have the same function field:

$$\mathbb{F}_q(X) = \mathbb{F}_q(\tilde{X}).$$

Diagram

Let Q be a point on X and let $P_1, \ldots, P_n$ be the points on $\tilde{X}$ such that $\nu(P_i) = Q$ for all $i = 1, \ldots, n$.

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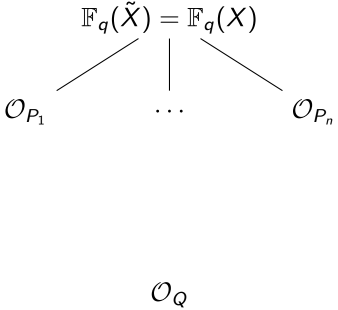

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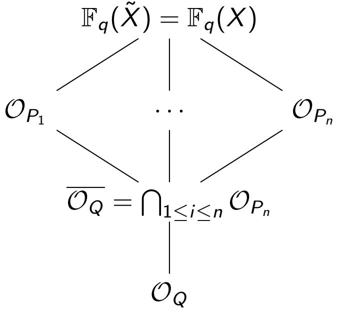

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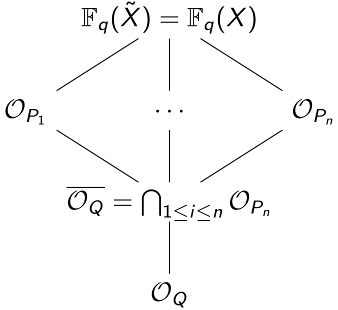

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The arithmetic genus

$\overline{\mathcal{O}_Q}/\mathcal{O}_Q$ is a finite dimensional $\mathbb{F}_q$-vectorial space. We set:

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We can now define the **arithmetic genus** π of a curve X as the integer:

$$\pi := g + \sum_{Q \in \operatorname{Sing} X(\overline{\mathbb{F}_q})} \delta_Q,$$

where g is the genus of the normalization $\tilde{X}$ of X (g is called the **geometric genus** of X).

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- $\pi \geq g$;
- $\pi = g$ if and only if X is a smooth curve;
- If X is a plane curve of degree d, $\pi = \frac{(d-1)(d-2)}{2}$.

Bounds for singular curves

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from which they obtain directly the equivalent of Serre-Weil bound for singular curves:

$$|\sharp X(\mathbb{F}_q) - (q+1)| \leq g[2\sqrt{q}] + \pi - g.$$

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Obviously we have

$$N_q(g,g) = N_q(g),$$

$$N_q(g,\pi) \leq N_q(g) + \pi - g$$

Fukasawa, Homma and Kim's curve

In 2011, Fukasawa, Homma and Kim consider and study the rational plane curve B over $\mathbb{F}_q$ defined by the image of

$$\begin{array}{cccc} \Phi: & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & (s,t) & \mapsto & (s^{q+1}, s^q t + s t^q, t^{q+1}) \end{array}$$

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$$\Downarrow$$

$$N(0, \frac{q^2-q}{2}) = N_q(0) + \frac{q^2-q}{2}$$

Question

Does there exist other different values of g and π for which

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To try to answer this question we need to find some way to construct singular curves with prescribed geometric genus g and arithmetic genus π and "many" rational points.

Singular curves with many points

Theorem

Let X be a smooth curve of genus g defined over $\mathbb{F}_q$. Let π be an integer of the form

$$\pi = g + a_2 + 2a_3 + 3a_4 + \cdots + (n-1)a_n$$

with $0 \leq a_i \leq B_i(X)$, where $B_i(X)$ is the number of closed points of degree i on the curve X. Then there exists a (singular) curve X' over $\mathbb{F}_q$ of arithmetic genus π such that X is the normalization of X' (so that X' has geometric genus g) and

$$\sharp X'(\mathbb{F}_q) = \sharp X(\mathbb{F}_q) + a_2 + a_3 + a_4 + \cdots + a_n.$$

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Roughly speaking we can "transform" a nonsingular point of degree d in a singular rational one provided that we increase the value of the arithmetic genus of $d - 1$.

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Let us take on the curve X:

- a_2 closed points of degree 2 : $S_2 = \{Q_1^{(2)}, Q_2^{(2)}, \dots, Q_{a_2}^{(2)}\}$;
- a_3 closed points of degree 3 : $S_3 = \{Q_1^{(3)}, Q_2^{(3)}, \dots, Q_{a_3}^{(3)}\}$;

  $\vdots$
- a_n closed points of degree n : $S_n = \{Q_1^{(n)}, Q_2^{(n)}, \dots, Q_{a_n}^{(n)}\}$;

$\Downarrow$

$S := S_2 \cup S_3 \cup \dots \cup S_n$.

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- $\mathcal{O}_Q$ is the integral closure of $\mathcal{O}'_Q$;
- $\mathcal{O}_Q/\mathcal{O}'_Q$ is an $\mathbb{F}_q$-vectorial space of dimension $\deg Q - 1$.

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By construction we obtain that:

- $\tilde{X}'$=X so that X' has geometric genus g;
- $\sharp X'(\mathbb{F}_q) = \sharp X(\mathbb{F}_q) + |S| = \sharp X(\mathbb{F}_q) + a_2 + a_3 + \cdots + a_n$;
- X' has arithmetic genus $\pi = g + a_2 + 2a_3 + 3a_4 + \cdots + (n-1)a_n$.

Remarks

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- this construction corresponds to a glueing of points on the curve obtained from X by extension of the base field to its algebraic closure.

Diagram

$\mathbb{F}_q(X)$

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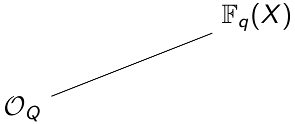

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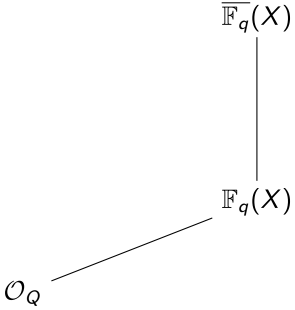

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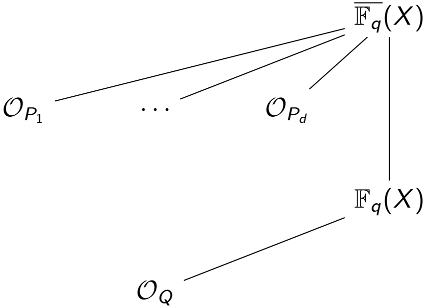

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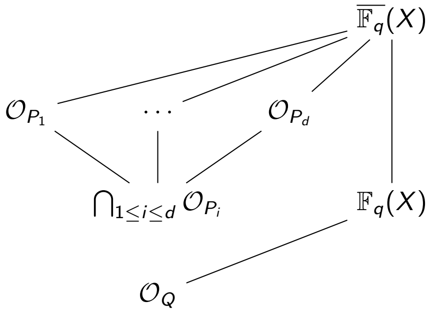

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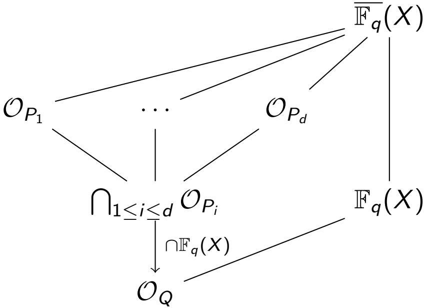
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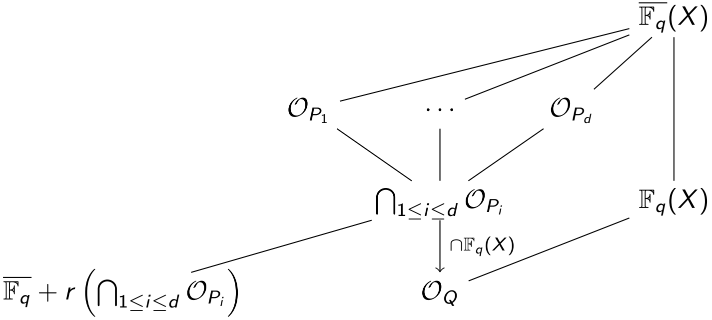
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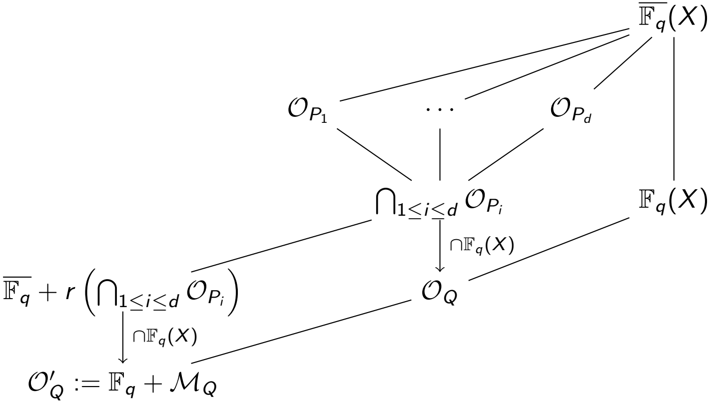
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As

$$B_2(\mathbb{P}^1) = \frac{q^2 - q}{2},$$

we have:

Proposition

For any $\pi \leq \frac{q^2-q}{2}$, there exists a (singular) rational curve X' over $\mathbb{F}_q$ of arithmetic genus π that attains the Aubry-Perret bound, i.e.

$$\sharp X(\mathbb{F}_q) = q + 1 + \pi.$$

In other terms we have

$$N_q(0, \pi) = N_q(0) + \pi = q + 1 + \pi.$$

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$$\sharp X(\mathbb{F}_q) = q + 1 + g[2\sqrt{q}] + \pi - g.$$

Proposition

If X is a maximal curve defined over $\mathbb{F}_q$ with q a square, of geometric genus g and arithmetic genus π, then:

$$2g(\sqrt{q} + q - 1) + 2\pi \leq q^2 - q.$$

Maximal curves

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In particular, for a maximal rational curve (and for q not necessarily square), this proposition implies:

$$\pi \leq \frac{q^2 - q}{2}$$

Proposition

We have

$$N_q(0, \pi) = q + 1 + \pi$$

if and only if $\pi \leq \frac{q^2 - q}{2}$.

With this proposition we completely answer the question when $g = 0$.

Thank you for the attention